Dimensional Regularization and Mellin Summation in High-Temperature Calculations

D. J. Bedingham

Theoretical Physics, The Blackett Laboratory, Imperial College,
Prince Consort Road, London, SW7 2BW, U. K.
E-mail: dj.bedingham@ic.ac.uk

Abstract

A general method for calculating asymptotic expansions of infinite sums in thermal field theory is presented. It is shown that the Mellin summation method works elegantly with dimensional regularization. A general result is derived for a class of one-loop Feynman diagrams at finite-temperature.

1 Method

The infinite sums often encountered in thermal Feynman diagrams are commonly computed using the function *coth*, or one with similar properties, to generate poles in the complex plane whose residues correspond to the terms in the sum [1]. This transforms the summation into a contour integration and conveniently splits the zero-temperature and thermal contributions. The method ceases to be ideal when calculating the high temperature asymptotic expansion of such sums. Cancellations occur between the thermal and non-thermal parts suggesting that the calculation could be streamlined. Here we propose a more concise method using dimensional regularization and the Mellin transform pair. The sums we shall consider occur in one-loop calculations and though these are well understood, the aim of this work is to outline a convenient and general method for their evaluation in the high temperature limit.

We recall first that the Mellin transform pair [2] can be written in the form

$$\mathcal{M}[f;s] = \int_0^\infty x^{s-1} f(x) dx, \tag{1}$$

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\mathcal{M}[f;s] ds. \tag{2}$$

The transform normally exists only in a strip $\alpha < \mathrm{Re}[s] < \beta$, and the inversion contour must lie in this strip: $\alpha < c < \beta$.

We will find that representing a function as in (2) is particularly useful for asymptotic analysis. We will look at infinite sums of the type

$$I = \frac{1}{\beta}\sum_{n\neq 0} f(\omega_n) \tag{3}$$

where

$$f(\omega_n) = \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{\omega_n^{2t}}{(\mathbf{p}^2 + \omega_n^2 + m^2)^\sigma} \tag{4}$$

and $\omega_n = 2\pi n/\beta$ for bosons (the method easily generalises to fermionic integrals). We exclude the $n = 0$ term of bosonic sums, however, as this may always be calculated independently.

By taking the Mellin transform of I with respect to ω_n, the sum can be represented using equations (1) and (2) as

$$\begin{aligned} I &= \frac{1}{2\pi i\beta} \sum_{n\neq 0} \int_{c-i\infty}^{c+i\infty} \left(\frac{2\pi n}{\beta}\right)^{-s} \mathcal{M}[f;s] ds & (5) \\ &= \frac{1}{\pi i\beta} \int_{c-i\infty}^{c+i\infty} \left(\frac{2\pi}{\beta}\right)^{-s} \zeta(s)\mathcal{M}[f;s] ds. & (6) \end{aligned}$$

Provided that $c > 1$, the interchange of the sum and integral is permitted by the uniform convergence of the sum with respect to s. The Mellin transform is not necessarily convergent for $s > 1$ and may require regulation. For example, one can use dimensional regularization and let d in (4) become small.

We now take a closer look at the Mellin transform of f. From the definition of a d-dimensional integral [3], we may write

$$\begin{aligned} \mathcal{M}[f;s] &= \int_0^\infty dy y^{s-1} \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{y^{2t}}{(\mathbf{p}^2 + y^2 + m^2)^\sigma} & (7) \\ &= \frac{\Gamma(s/2+t)}{2\pi^{s/2+t}} \int d^{s+2t}\mathbf{y} \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{1}{(\mathbf{p}^2 + \mathbf{y}^2 + m^2)^\sigma}. & (8) \end{aligned}$$

We may further combine these integrations to give

$$\mathcal{M}[f;s] = \frac{\Gamma(s/2+t)}{2\pi^{s/2+t}} \int \frac{d^{d+s+2t}\mathbf{p}}{(2\pi)^d} \frac{1}{(\mathbf{p}^2 + m^2)^\sigma}. \tag{9}$$

We now see why the Mellin summation technique is particularly suited to this type of calculation. The Mellin transform neatly combines with the d-dimensional integral leaving a $(d+s+2t)$-dimensional integral which can easily be evaluated in terms of Γ-functions:

$$\mathcal{M}[f;s] = \frac{\Gamma(s/2+t)}{2(4\pi)^{d/2}} \frac{\Gamma(\sigma - d/2 - s/2 - t)}{m^{2\sigma-d-s-2t}\Gamma(\sigma)}. \tag{10}$$

With $t, \sigma > 0$, the integral is convergent when $d < 2(\sigma - t) - s$ and so we must choose $2(\sigma - t) - d > c > 1$. This can always be achieved by dimensional regularization.

Combining equations (6) and (10) we have

$$\begin{aligned} I &= \frac{m^{-2\sigma+d+2t}}{\Gamma(\sigma)(4\pi)^{d/2}\beta} \frac{1}{2\pi i} \times \\ &\times \int_{c-i\infty}^{c+i\infty} \left(\frac{2\pi}{m\beta}\right)^{-s} \zeta(s)\Gamma(s/2+t)\Gamma(\sigma - d/2 - s/2 - t) ds. & (11) \end{aligned}$$

From the asymptotic behaviour of the integrand we conclude that we may close the integration path at infinity in the positive real half-plane. The only poles which are enclosed within the contour are those due to $\Gamma(\sigma - d/2 - s/2 - t)$. This has simple poles at $s = 2(\sigma - t) - d + 2k$ for $k = 0, 1, 2 \ldots \infty$.

Evaluating the residues we find

$$
\begin{aligned}
I &= \frac{2(4\pi)^{-d/2}}{\Gamma(\sigma)\beta}\left(\frac{2\pi}{\beta}\right)^{d-2(\sigma-t)} \times \\
&\quad \times \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta\left(2(k+\sigma-t)-d\right)\Gamma(k+\sigma-d/2)\left(\frac{m\beta}{2\pi}\right)^{2k} \qquad (12)
\end{aligned}
$$

which is our general result.

2 Example: Scalar Tag Diagram

Consider as a specific example the sum

$$
J = \frac{\mu^{2\epsilon}}{\beta}\sum_{n\neq 0}\int\frac{d^{3-2\epsilon}\mathbf{q}}{(2\pi)^{3-2\epsilon}}\frac{1}{(\mathbf{q}^2+\omega_n^2+m^2)} \qquad (13)
$$

which is the tag diagram of scalar field theory. We may use equation (12) with $\sigma = 1, t = 0$ and $d = 3 - 2\epsilon$, to obtain

$$
\begin{aligned}
J &= \frac{1}{(4\pi)^{1/2}\beta^2}\left(\frac{\mu^2\beta^2}{\pi}\right)^{\epsilon} \times \\
&\quad \times \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\zeta(2k-1+2\epsilon)\Gamma(k-1/2+\epsilon)\left(\frac{m\beta}{2\pi}\right)^{2k}. \qquad (14)
\end{aligned}
$$

As we let ϵ tend to zero all terms in this series are well defined except for the $k = 1$ term. This term has a pole in the ζ-function corresponding to the zero-temperature divergence. For small ϵ we have

$$
\begin{aligned}
\zeta(1+2\epsilon) &= \frac{1}{2\epsilon}+\gamma+\mathcal{O}(\epsilon) & (15)\\
\Gamma(1/2+\epsilon) &= \Gamma(1/2)\left[1-\epsilon(2\ln 2+\gamma)+\mathcal{O}(\epsilon^2)\right] & (16)\\
\left(\frac{\mu^2\beta^2}{\pi}\right)^{\epsilon} &= 1+\epsilon\ln\frac{\mu^2\beta^2}{\pi}+\mathcal{O}(\epsilon^2) & (17)
\end{aligned}
$$

where γ is the Euler-Mascheroni constant, resulting in

$$
J_{k=1} = \frac{m^2}{(4\pi)^2}\left[\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi\right)+2\ln\frac{\mu\beta e^{\gamma}}{4\pi}\right]+\mathcal{O}(\epsilon). \qquad (18)
$$

The first few terms in the series are then given by

$$
\begin{aligned}
J &= \frac{1}{12\beta^2}+\frac{m^2}{(4\pi)^2}\left[\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi\right)+2\ln\frac{\mu\beta e^{\gamma}}{4\pi}\right] \\
&\quad +\frac{2(m^2)^2\beta^2\zeta(3)}{(4\pi)^4}+\mathcal{O}(\beta^4). \qquad (19)
\end{aligned}
$$

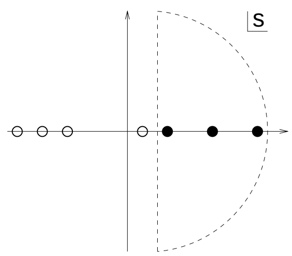

Figure 1: The Mellin inversion contour is closed in the positive real half plane. The only poles enclosed by this contour are the UV divergences of the Mellin transform (filled dots). Poles external to the contour include that of the ζ-function at $s = 1$ and those due to infrared divergences in the Mellin transform.

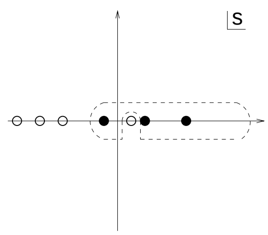

Figure 2: As the regulator is removed, the poles move to the left. A pole inside the contour may tend to coincide with ζ-function pole - this will generate the zero-temperature divergence.

3 Summary

We may summarise the method in a prescription for generating the high temperature expansion of one-loop thermal Feynman diagrams.

(i) We express the sum as in equation (6). The Mellin transform must be regulated by suitable choice of dimension, d, such that it exists for $1 < \mathrm{Re}[s]$. The inversion integration can then be defined with $1 < c$.

(ii) The contour is closed in the right-hand half-plane (see figure 1). In order to know if this arc gives a contribution to the sum we should check the asymptotic value of the integrand of (6) as $|s| \to \infty$.

(iii) We then deform the contour as we allow our regulator d to tend to its original value (figure 2). The result is the same if we first calculate the residues and then reinstate d.

We may anticipate divergences resulting from pinch singularities between poles inside and outside the contour. These will correspond to the zero-temperature divergences and are also responsible for generating the $\ln \mu\beta$ terms as seen in the example ($\ln \mu/m$ terms do not appear at high temperature). In general, it is seen that contributions to the sum result from poles in s of the $(d + s + 2t)$-dimensional integral of equation (9).

Acknowledgements

I am grateful to Tim Evans for his input of ideas and guidance.

References

[1] J. I. Kapusta, Finite-Temperature Field Theory (Cambridge Univ. Press, Cambridge 1989)

[2] B. Davis, Integral Transforms and their Applications (Springer-Verlag, New York 1985)

[3] J. Collins, Renormalization (Cambridge Univ. Press, Cambridge 1984)